ON EXTENSION PROPERTY OF CANTOR-TYPE SETS

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Abstract. We suggest a new approach to prove the dominating norm property for spaces $\mathcal{E}(K)$ of Whitney functions, based on the estimation of least deviation of polynomials on Cantor-type sets. In this way we prove that the generalized Cantor sets of finite type and logarithmic dimension 1 have the extension property, since by Tidten-Vogt characterization a compact set K has the extension property iff the space $\mathcal{E}(K)$ has the property DN.

1. Introduction

Let K be a compact set without isolated points in $\mathbb{R}$. Then $\mathcal{E}(K)$ is the space of Whitney functions with the topology defined by the norms

$$\| f \|_q = |f|_q + sup\left\{ \frac{|(R_y^q f)^{(k)}(x)|}{|x-y|^{q-k}} : x, y \in K, x \neq y, k = 0, 1, ...q \right\},$$

$q = 0, 1, ...,$ where $|f|_q = sup\{|f^{(k)}(x)| : x \in K, k \leq q\}$ and $R_y^q f(x) = f(x) - T_y^q f(x)$ is the Taylor remainder. We say that K has the extension property if there exists a linear continuous extension operator $L : \mathcal{E}(K) \to C^\infty(\mathbb{R})$. The problem of geometric characterization of extension property goes back to Mityagin [4]. In [1] it was proved that the generalized Cantor sets of finite type with logarithmic dimension > 1 (see [1] for definitions and details; see [3] for the bibliography) have the extension property, whereas for the case with logarithmic dimension <1 this is no longer true. Here we consider model Cantor-type sets of logarithmic dimension 1 and show that they have the extension property. The proof is based on the estimation of least deviation for polynomials on Cantor-type sets.

2. Dominating Norm Property.

We shall use the property DN (see [7]) of Fréchet spaces, which can be given as follows (see e.g. [3],[1]):

1991 *Mathematics Subject Classification.* Primary 46E10; Secondary 26C10.

$$\exists p\,, \exists R>0:\ \forall\, q\, \exists\, r, C:\ \|\cdot\|_q \le t^R \|\ \cdot\ \|_p + \frac{C}{t}\|\cdot\|_r,\ t>0. \qquad (1)$$

Here and in the sequel we suppose that the system of seminorms of Fréchet space is increasing; $p, q, r \in \mathbb{N}_0 := \{0, 1, ...\}$.

Due to Tidten ([5], Folg.2.4) we have the following characterization: a compact set K has the extension property iff the space $\mathcal{E}(K)$ has the property DN. Due to Tidten and Frerick (see e.g Lemma 1 in [6]) in the case of spaces of Whitney functions one can replace the norm $\|\cdot\|_q$ in (1) by simple sup-norm $|\cdot|_q$. Obviously, it suffices to consider only elements of increasing sequence (q_v). Thus, in order to show the extension property of a compact set K it is enough to prove that

$$\exists R>0:\ \forall\, q=2^v\ \exists\, r,\ C,\ t_0:\ \ \forall t>t_0,\, \forall f\in\mathcal{E}(K)$$

$$|f|_0 \le t^{-R},\ \ \|f\|_r \le t \Longrightarrow |f|_q \le C. \qquad (2)$$

3. **Estimation of least deviation for polynomials on Cantor-type sets.**

Let $N \ge 2$ be integer and $(l_n)_{n=0}^{\infty}$ be a sequence such that $l_0 = 1$, $0 < N \cdot l_n < l_{n-1}\,,\, n \in \mathbb{N}$. Let K_N be the Cantor set associated with the sequence (l_n) that is $K = \bigcap_{n=0}^{\infty} E_n$, where $E_0 = I_{0,1} = [0, 1]$, E_n is a union of N^n closed *basic* intervals $I_{n,k}$ of length l_n and E_{n+1} is obtained by deleting of $N-1$ open equidistant subinterval of length h_{n+1}, $h_{n+1} = \frac{l_n - N\, l_{n+1}}{N-1}$, from each $I_{n,k}\ , k = 1, 2, ...N^n$.

Given sequence $(\alpha_n)_{n=2}^{\infty}$ let us denote by $K_N^{(\alpha_n)}$ the Cantor set associated with the sequence (l_n), where $l_0 = 1$, $l_1 < 1/N$ and $l_n = l_{n-1}^{\alpha_n} = ... = l_1^{\alpha_2\cdots\alpha_n}, n \ge 2$. Here we consider only the case $\alpha_n \to N$, which gives the compact sets with logarithmic dimension 1, so we can suppose that the first elements of the sequence (α_n) are chosen in such a way that the compact set $K_N^{(\alpha_n)}$ is well-defined. Also without loss of generality we can restrict ourselves by condition

$$l_n \le h_n,\ \forall n. \qquad (3)$$

Given $m \in \mathbb{N}$ and a compact set K we will consider the value of least deviation $\Delta_m(K) = \inf_{P\in\Pi'_m} \sup_{z\in K} |P(z)|$, where Π'_m stands for the set of all polynomials of degree less than or equal to m with the leading coefficient equal to 1.

Lemma 1. *Given integer $N \ge 2$ let $K_N = \cup_1^N I_k$ be a union of equidistant intervals I_k of length l with $l \le h$, where h is the distance between neighboring intervals. Then $\Delta_N(K_N) \ge l/2 \cdot h^{N-1}$.*

This follows by de la Valée Poussin's Theorem (see e.g. [2], T.5.2). We see that some zeros of the polynomial of least deviation on K_N do not belong to the compact set already for $N \ge 4$ provided that the length l is sufficiently small.

Lemma 2. *Let K_N be a Cantor-type compact set associated with the sequence (l_n). Given $v, n \in \mathbb{N}$ and m with $N^{v-1} < m \le N^v$ let us take any basic interval $I_{n,k}$, $k = 1, ..., N^n$, of the compact set K_N. Then*

$$\Delta_m(K_N \cap I_{n,k}) \ge (4N)^{-N^v} l_{n+v} l_{n+v-1}^{N-1} l_{n+v-2}^{(N-1)N} \cdots l_n^{(N-1)N^{v-1}}.$$

Proof: Set $K = K_N \cap I_{n,k}$. Since the values $\Delta_m(K)$ do not increase, it is enough to show the inequality only for $m = N^v$. We proceed by induction on v. The case $v = 1$ is given by Lemma 1. Suppose that the desired inequality holds for the value $v-1$. The interval $I_{n,k}$ covers N intervals $I_{n+1,j}$, $j = 1, 2, ..., N$. Consider the circles $B_j = \{z \in \mathbb{C} : |z - c_j| < \frac{1}{2}(l_{n+1} + h_{n+1})\}$, $j = 1, 2, ..., N$, where c_j is the midpoint of $I_{n+1,j}$. Let Q_m, $Q_m(x) = \prod_{i=1}^m (x - \zeta_i)$, be the polynomial of least deviation on K. Let k_j be the number of zeros of Q_m in the circle B_j, $j = 1, 2, ..., N$. Clearly, there exists a number j_0 such that $k_{j_0} \le N^{v-1}$.

Suppose at first that $k_{j_0} = N^{v-1}$. Then for any alternation point a of the polynomial $Q_{N^{v-1}}$ of least deviation on $L := K \cap I_{n+1,j_0}$ we get

$$\Delta_m(K) \ge |Q_m(a)| \ge \Delta_{N^{v-1}}(L) \cdot \prod_{\zeta_i \notin B_{j_0}} |a - \zeta_i|$$

$$\ge (4N)^{-N^{v-1}} l_{n+v} l_{n+v-1}^{N-1} l_{n+v-2}^{(N-1)N} \cdots l_{n+1}^{(N-1)N^{v-2}} \cdot (l_n/4N)^{N^v - N^{v-1}},$$

as $|a - \zeta_i| > h_{n+1}/2 > \frac{1}{4N} l_n$ by (3) for $\zeta_i \notin B_{j_0}$.

Now let $k_{j_0} < N^{v-1}$. Then we can take any $N^{v-1} - k_{j_0}$ zeros of Q_m from the outside the circle B_{j_0} and place them arbitrarily on L. Let us denote by $\tilde{Q}_m$ the polynomial obtained after this procedure. Then for any point $a \in L$ we get the bound $|Q_m(a)| \ge |\tilde{Q}_m(a)|$ and one can apply the previous arguments to the polynomial $\tilde{Q}_m$. □

Thus, in the case of compact set $K_N^{(\alpha_n)}$ we have the bound

$$\Delta_m(K_N^{(\alpha_n)} \cap I_{n,k}) \ge (4N)^{-N^v} l_n^{\omega(v,n,N)}$$

with

$$\omega(v, n, N) = (N-1)N^{v-1} + \alpha_{n+1}(N-1)N^{v-2} + \cdots$$
$$+ \alpha_{n+1} \cdots \alpha_{n+v-1}(N-1) + \alpha_{n+1} \cdots \alpha_{n+v}. \quad (4)$$

Lemma 3. *Given fixed natural s let v, m be natural numbers with $1 \le v \le s$, $N^{v-1} < m \le N^v$. Let $\omega = \omega(v, n, N)$ be given by (4), where $\alpha_n \to N$. Then there exists $n_0 = n_0(s)$ such that for all $n \ge n_0$ we have $\omega < m[(N-1)v + N]$ and $m > \frac{\omega}{(N-1)\cdot \log_N(N^3\omega)}$.*

The proof is straightforward.

For simplicity in what follows we consider the case $N = 2$, since the general case is quite similar. So, if $\alpha_n \to 2$, $1 \le v \le s$ and $2^{v-1} < m \le 2^v$, then for any basic interval $I_{n,k}$ with sufficiently large n we get the bound $\Delta_m(K_2^{(\alpha_n)} \cap I_{n,k}) \ge \delta_s l_n^\omega$. Here δ_s is positive and depends only on s and

$$\omega < m(v+2), \quad m > \frac{\omega}{\log_2(8\omega)}. \quad (5)$$

4. Extension property of $K_2^{(\alpha_n)}$, $\alpha_n \to 2$.

Theorem 1. *Let $\alpha_n \to 2$. Then the space $\mathcal{E}(K_2^{(\alpha_n)})$ has the Dominating Norm property.*

Proof: We can take any $R \geq 15$. Given $q = 2^v$ (let $v \geq 6$) take $u = p \cdot v$ with arbitrary natural $p \geq 5$ and $r = 2^s$, where $s = (p+2)v$.

Let $\sigma_s = \delta_s \, l_s \, l_{s-1} \cdots l_0^{2^{s-1}}$, $t_0 = 2^{r+1}\sigma_s^{-1} r!$. Fix $t \geq t_0$ and $f \in \mathcal{E}(K_2^{(\alpha_n)})$ such that $|f|_0 \leq t^{-R}$, $\|f\|_r \leq t$. We want to show (2), that is

$$|f^{(i)}(y)| \leq C_q, \;\; i \leq q, \;\; y \in K_2^{(\alpha_n)},$$

where C_q does not depend on t, f, y. Let us fix $y \in K_2^{(\alpha_n)}$. There is no loss of generality in assuming that $y = 0$. We will denote by P the $r-$th Taylor polynomial of f at $x = 0$:

$$P(x) = T_0^r f(x) = \frac{f^{(m)}(0)}{m!} \prod_{j=1}^{m} (x - \zeta_j).$$

Here m is the maximal number such that $m \leq r$ and $f^{(m)}(0) \neq 0$, $\zeta_j \in \mathbb{C}$ with $|\zeta_j| \leq |\zeta_{j+1}|$, $j = 1, 2, \cdots, m-1$. Since $|R_0^r f(x)| = |f(x) - P(x)| \leq t\, x^r$, then $|P(x)| \leq t^{-R} + t\, x^r$ for any $x \in K_2^{(\alpha_n)}$. Fix $x_t = t^{-\frac{R+1}{r}}$ and $n : \; l_n \leq x_t < l_{n-1}$. We can assume that for all indexes larger than this n one can use the bound (5), since otherwise we replace t_0 by the larger one. Also we suppose that for any $l \geq n$ and $w \leq s$ the product $\alpha_{l+1}\alpha_{l+2}\cdots\alpha_{l+w}$ does not exceed 2^{w+1}.

Clearly, $|P(x)| \leq 2t^{-R}$ for $x \in K_2^{(\alpha_n)} \cap [0, l_n]$. The basic idea is to show that the number of zeros of P near the origin is rather large.

The $i-$th derivative of P represents the sum of $\frac{m!}{(m-i)!}$ products where every product contains $m - i$ terms of the type $(x - \zeta_j)$. Therefore

$$|f^{(i)}(0)| = |P^{(i)}(0)| \leq \frac{|f^{(m)}(0)|}{(m-i)!} \prod_{j=i+1}^{m} |\zeta_j|.$$

Let i_0 be such that $\frac{1}{(m-i_0)!} \prod_{j=i_0+1}^{m} |\zeta_j| = \max_{i \leq q} \frac{1}{(m-i)!} \prod_{j=i+1}^{m} |\zeta_j|$.

Let $M = max\{j : |\zeta_j| \leq 2\}$ be the number of "not large" roots of P. Let $m_k = max\{j : |\zeta_j| \leq 2\, l_{n+k}\}$, $k = 0, 1, \cdots, u$. Clearly, $m_u \leq m_{u-1} \leq \cdots \leq m_0 \leq M \leq m$.

We now decompose the proof in a few steps.

1. Below bound for m_u.

Let us show that we can suppose $m_u \geq q$. In fact, if $m_u < q$, then let $\nu = \max\{m_u, i_0\}$, $Q(x) = \prod_{j=1}^{\nu}(x - \zeta_j)$. Of course, $\nu \leq q$. Therefore there exists $z \in K_2^{(\alpha_n)} \cap [0, l_{n+u}]$ such that $|Q(z)| \geq \Delta_\nu(K_N^{(\alpha_n)} \cap [0, \, l_{n+u}]) \geq \Delta_q(K_N^{(\alpha_n)} \cap [0, \, l_{n+u}]) \geq \delta_s \; l_{n+u}^{\omega(v, \, n+u, \, 2)}$, by (5). Then

$$|Q(z)| \geq \delta_s l_{n-1}^{\alpha_n \alpha_{n+1} \cdots \alpha_{n+u} q(v+2)} \geq \delta_s x_t^{2^{u+2} q(v+2)}.$$

Now $|P(z)| = \frac{|f^{(m)}(0)|}{m!}\,|Q(z)|\prod_{j=\nu+1}^m |z-\zeta_j|$. Since $|\zeta_j| > 2l_{n+u}$ for $j \geq \nu+1$, then $z \leq l_{n+u} \leq |\zeta_j| - z \leq |\zeta_j - z|$ and $|\zeta_j| \leq |\zeta_j - z| + z \leq 2\,|\zeta_j - z|$. Then

$$2\,t^{-R} \geq |P(z)| \geq \frac{|f^{(m)}(0)|}{m!}\prod_{j=\nu+1}^m |\zeta_j| \cdot (1/2)^{m-\nu}|Q(z)|$$

and $|f^{(m)}(0)|\prod_{j=\nu+1}^m |\zeta_j| \leq 2^{m+1-\nu}\,m!\;t^{-R}\delta_s^{-1}\,x_t^{-2^{u+2}q(v+2)}$. Note also that $\prod_{j=i_0+1}^m |\zeta_j| \leq \prod_{j=\nu+1}^m |\zeta_j|$. In fact, this is trivial if $\nu = i_0$. Otherwise, $|\zeta_j| \leq 2\,l_{n+u} < 1$ for $i_0 < j \leq \nu$. From here we get $|f^{(m)}(0)|\prod_{j=i_0+1}^m |\zeta_j| \leq C\,t^{\mu}$, where $\mu = \frac{R+1}{r}\;2^{u+2}\,q(v+2) - R$ and the constant C depends only on q, r. Applying $R+1 < 2R$, we estimate μ from above:

$$\mu < R \cdot 2^{3-s+u+v}(v+2) - R \leq 0,$$

as $8(v+2) \leq 2^v$ due to the choice of v. Thus for $m_u < q$ we get the desired bound $\max_{i\leq q}|P^{(i)}(0)| \leq C$ and we can restrict ourselves by the case $m_u \geq q$. In addition, this means that $i_0 = q$. Thus we only need to show that

$$|f^{(m)}(0)|\prod_{j=q+1}^m |\zeta_j| \leq C, \tag{6}$$

where the constant C depends only on q, r.

2. Representation of the product of large roots.

Let us take $\lambda = \lambda(f,t)$ such that $|f^{(m)}(0)|\prod_{|\zeta_j|>2}|\zeta_j| = t^{\lambda}$. Here and in the sequel $\prod_{\emptyset} = 1$. We want to show that $0 < \lambda < 2$. In fact, if $|\zeta_j| \leq 2,\ \forall j$, then

$$t^{\lambda} = |f^{(m)}(0)| \leq |f|_r \leq t$$

and $\lambda \leq 1$. If $|\zeta_j| > 2$ for some j, then we take $Q(x) = \prod_{j=1}^M (x - \zeta_j)$. Since $M \leq r$, then by Lemma 2 there exists $z \in K_2^{(\alpha_n)}$ such that $|Q(z)| \geq \sigma_s$. For any ζ_j with $|\zeta_j| > 2$ we get as before $|z - \zeta_j| \geq 1/2\,|\zeta_j|$. Therefore, $\prod_{|\zeta_j|>2}|z-\zeta_j| \geq (1/2)^r\prod_{j=M+1}^m |\zeta_j|$.

On the other hand, $|P(z)| \leq t^{-R} + tz^r < 2t$, so $2t > \sigma_s 2^{-r}\,\frac{1}{m!}\,t^{\lambda}$ and $\lambda < 2$, as $t \geq t_0$.

Note also that if $\lambda \leq 0$, then

$$|f^{(m)}(0)|\prod_{j=q+1}^m |\zeta_j| \leq \prod_{j=q+1}^M |\zeta_j| \leq 2^{M-q} < 2^r,$$

so we can exclude this case as well.

3. Below bound for m_k.

We now use the same method as in 1 in order to estimate m_k from below in terms of r. Fix k from $\{0, 1, \cdots, u\}$ and v_k with $2^{v_k - 1} < m_k \leq 2^{v_k}$. Let $Q(x) = \prod_{j=1}^{m_k}(x - \zeta_j)$. Then there exists $z \in K_2^{(\alpha_n)} \cap [0, l_{n+k}]$ such that $|Q(z)| \geq \delta_s l_{n+k}^{\omega(v_k,\,n+k,\,2)}$, where $\omega(v_k,\,n+k,\,2)$ is given by (4). Since

$2\,t^{-R} \geq |P(z)| = \frac{|f^{(m)}(0)|}{m!} \prod_{j=m_k+1}^{m} |z-\zeta_j| \cdot |Q(z)|$ and $|z-\zeta_j| \geq 1/2\ |\zeta_j|$ for $j \geq m_k+1$, so

$$|f^{(m)}(0)| \prod_{j=m_k+1}^{m} |\zeta_j| \leq 2^{m+1-m_k}\ m!\ t^{-R}\ |Q(z)|^{-1}.$$

Now $\prod_{j=q+1}^{m} |\zeta_j| = \prod_{j=q+1}^{m_k} |\zeta_j|\ \prod_{j=m_k+1}^{m} |\zeta_j|$, as $m_k \geq q$. Therefore,

$$|f^{(m)}(0)| \prod_{j=q+1}^{m} |\zeta_j| \leq 2^{m+1-m_k}\ m!\ t^{-R} \prod_{j=q+1}^{m_k} |\zeta_j|\ |Q(z)|^{-1}.$$

Notice that $|\zeta_j| \leq 2\,l_{n+k}$ for $j \leq m_k$. Using the bound for $|Q(z)|$, we get

$$|f^{(m)}(0)| \prod_{j=q+1}^{m} |\zeta_j| \leq C\,t^{-R}\,l_{n+k}^{-\omega(v_k,\,n+k,\,2)+m_k-q},$$

where the constant C depends only on q, r.

If $\omega(v_k,\,n+k,\,2)+q \leq m_k$, then we get the desired bound (6). So let us suppose that $\omega(v_k,\,n+k,\,2)+q > m_k$. Then $l_{n+k} = l_{n-1}^{\alpha_n\cdots\alpha_{n+k}} > x_t^{\alpha_n\cdots\alpha_{n+k}}$ and $|f^{(m)}(0)| \prod_{j=q+1}^{m} |\zeta_j| < C\,t^{\mu_k}$, where $\mu_k = \frac{R+1}{r}\,\alpha_n\cdots\alpha_{n+k}\,[\omega(v_k,\,n+k,\,2) - m_k+q] - R$. Again we can suppose that $\mu_k > 0$. Then $\omega(v_k,\,n+k,\,2) - m_k > \frac{R}{R+1}\,\frac{r}{\alpha_n\cdots\alpha_{n+k}} - q$ and $\omega(v_k,\,n+k,\,2) > \frac{R}{R+1}\,\frac{r}{\alpha_n\cdots\alpha_{n+k}}$, as $m_k \geq q$. Now by (5) we obtain $m_k > \frac{R}{R+1}\,\frac{r}{\alpha_n\cdots\alpha_{n+k}} \div [\log_2 \frac{8R}{R+1} + s - \log_2(\alpha_n\cdots\alpha_{n+k})]$. Since $\alpha_n\cdots\alpha_{n+k} > 2^k$ we get

$$m_k > \frac{R}{R+1} \cdot \frac{r}{\alpha_n\cdots\alpha_{n+k}} \cdot \frac{1}{s+3-k}. \tag{7}$$

4. Decomposition of the total product.

Finally,

$$|f^{(m)}(0)| \prod_{j=q+1}^{m} |\zeta_j| \leq |f^{(m)}(0)| \prod_{j=M+1}^{m} |\zeta_j| \prod_{j=q+1}^{m_u} |\zeta_j| \cdots \prod_{j=m_1+1}^{m_0} |\zeta_j| \prod_{j=m_0+1}^{M} |\zeta_j| \leq$$

$$t^{\lambda}\,(2\,l_{n+u})^{m_u-q} \cdot (2\,l_{n+u-1})^{m_{u-1}-m_u} \cdots (2\,l_{n+1})^{m_1-m_2} \cdot (2\,l_n)^{m_0-m_1} \cdot 2^{M-m_0}$$

$= 2^{M-q}\,t^{\lambda}\,l_n^{\Omega}$ with $\Omega = m_0 - m_1 + \alpha_{n+1}(m_1-m_2) + \alpha_{n+1}\alpha_{n+2}(m_2-m_3) + \cdots + \alpha_{n+1}\cdots\alpha_{n+u-1}(m_{u-1}-m_u) + \alpha_{n+1}\cdots\alpha_{n+u}(m_u-q)$.

Since $l_n \leq x_t$, we get $|f^{(m)}(0)| \prod_{j=q+1}^{m} |\zeta_j| \leq 2^r\,t^{\lambda-\frac{R+1}{r}\Omega}$. Thus, it is enough to show that

$$(R+1)\,\Omega \geq r\ \lambda. \tag{8}$$

Clearly, $\Omega = m_0 + (\alpha_{n+1}-1)m_1 + \alpha_{n+1}(\alpha_{n+2}-1)m_2 + \cdots + \alpha_{n+1}\cdots\alpha_{n+u-1}(\alpha_{n+u}-1)m_u - \alpha_{n+1}\cdots\alpha_{n+u}\,q$.

For any k, $k \in \{1, \cdots, u\}$ by (7) we obtain

$$\alpha_{n+1}\cdots\alpha_{n+k-1}(\alpha_{n+k}-1)m_k > \frac{1}{5}\ \frac{R}{R+1}\ \frac{r}{s+3-k},$$

since we consider the values α_n rather close to 2, so $\frac{\alpha_{n+k}-1}{\alpha_n\alpha_{n+k}} > \frac{1}{5}$. For $k=0$ we immediately get from (7) the bound $m_0 > \frac{1}{5}\ \frac{R}{R+1}\ \frac{r}{s+3}$.

On the other hand, $\alpha_{n+1} \cdots \alpha_{n+u}\, q < 2^{u+1} q = 2^{u+v+1}$.
Therefore,

$$(R+1)\,\Omega > \frac{R}{5}\, r \sum_{k=0}^{u} \frac{1}{s+3-k} - (R+1)\, 2^{u+v+1}.$$

But $\sum_{j=1}^{J} \frac{1}{s-j} > \int_0^J \frac{dx}{s-x} = \ln \frac{s}{s-J}$. Hence, $\sum_{k=0}^{u} \frac{1}{s+3-k} > \ln \frac{s+4}{s+3-u}$. The argument of logarithmic function here is larger than 3 due to the choice of u and s. Therefore, $(R+1)\,\Omega > \frac{R}{5} r - (R+1)\, 2^{u+v+1}$, which exceeds $r\, \lambda$, as is easy to check, and (8) is proved. □

5. **Conjecture.**

Clearly, the logarithmic dimension, which is a global characteristic of a set, can not in general give a geometric characterization of the local extension property. Already in the class of generalized Cantor sets of infinite type one can easily find compact sets of logarithmic dimension 1 without the extension property.

Example. Take $\alpha_n = N_n \uparrow \infty$. Then the corresponding Cantor-type set $K^{(\alpha_n)}_{(N_n)}$ has the logarithmic dimension 1, whereas the space $\mathcal{E}(K^{(\alpha_n)}_{(N_n)})$ has not the Dominating Norm property (see [1] for details).
For a geometric characterization of the extension property we suggest here the following "density of capacity" of a set.

Denote by ψ the function $\psi(r) = \ln^{-1} 1/r$, $0 < r < 1$, corresponding to the logarithmic measure. Given compact subsets K of $\mathbb{R}$ let $m_\delta(K)$ be the outer Housdorff measure of K with respect to the function $\psi^{1-\delta}$, $0 < \delta < 1$. For $x \in K$ let $\rho_\delta(x)$ be the lower density of the set K at the point x with respect to the measure m_δ, that is

$$\rho_\delta(x) = \liminf_{r \to 0} \frac{m_\delta(K \cap [x-r, x+r])}{m_\delta([x-r, x+r])}.$$

Conjecture. The compact set $K \subset \mathbb{R}$ has the extension property if and only if $inf_{x\in K}\ \rho_\delta(x) > 0$ for any $\delta \in (0,1)$.

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